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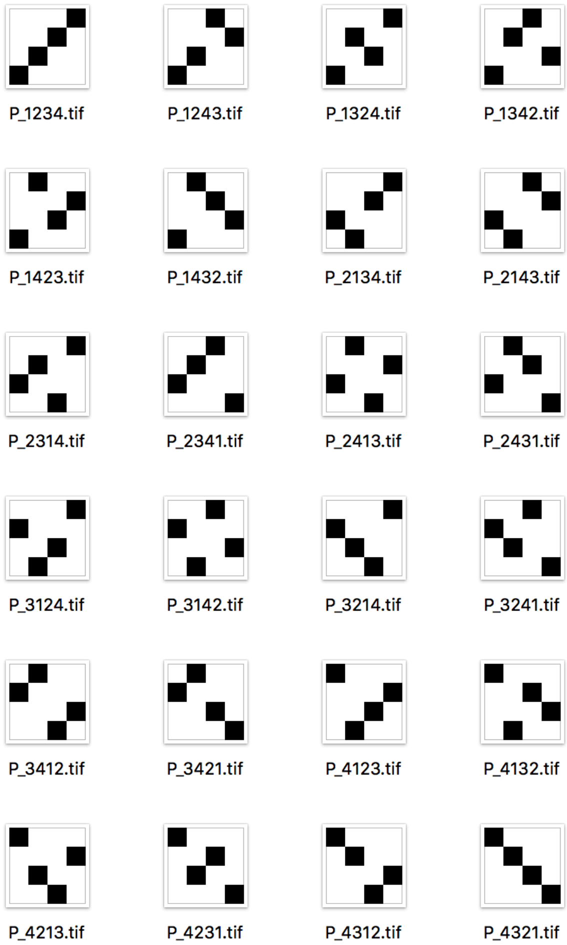
Permutations play an important role in the technique of weaving, whether for the declination of block graphics, for amalgamation or for telescoping.

This article specifically studies the permutations of 4 elements, and its results will be used in other articles.



A permutation is a bijection, that is, a diagram that has one and only one black box per row and column.

In mathematics it is a function that changes the order of the elements of a set.

In weaving it is a diagram that will allow us, for example, to change the order of the shafts of a tuck.

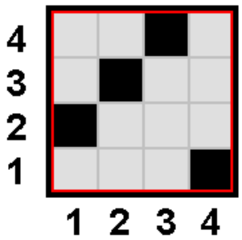
We will only focus in this article on the permutations of 4 elements.

For example, consider the following permutation :

(1, 2, 3, 4) old order
P = (2, 3, 4, 1) new order

Or more simply :
P = (2, 3, 4, 1)

Graphically we can represent this permutation as :

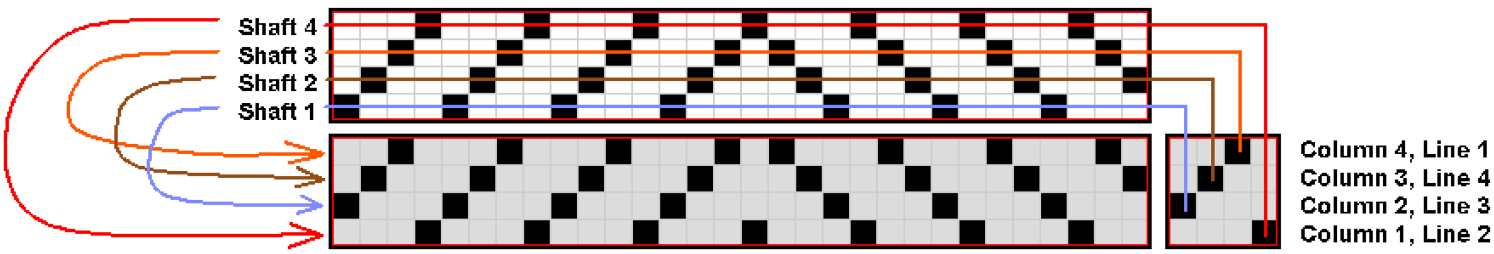


The lines are numbered from bottom to top.
The black pixels are listed from column 1 on the left, and column 4 on the right:

Column 1, Line 2
Column 2, Line 3
Column 3, Line 4
Column 4, Line 1

P = (2, 3, 4, 1)

Let's see how we can use this permutation to mix the shafts of a threading.



The peg-plan P is the permutation P = (2, 3, 4, 1)
The threading A is the diagram whose lines will be mixed.

Le diagramme de tissu est le résultat du calcul T = P o A

P o A est le rentrage A mélangé par P

Pick 1 of drawdown P o A = Shaft 4 of A
Pick 2 of drawdown P o A = Shaft 1 of A
Pick 3 of drawdown P o A = Shaft 2 of A
Pick 4 of drawdown P o A = Shaft 3 of A

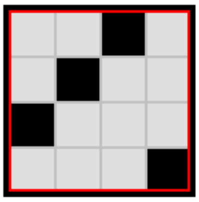
The drawdown formula coincides with the calculation of the permutations.

If we apply successively two permutations, first P = (2, 3, 4, 1)
then Q = (4, 3, 1, 2), we speak of the compound (or the multiplied) Q o P of permutations P and Q.

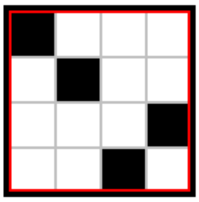
We can talk about the multiplication of permutations but we must be aware that the multiplication of permutations is not commutative as is the multiplication of numbers.

For permutations in general Q o P ≠ P o Q
Whereas for all the numbers a x b = b x a

Let's compound these two permutations :

Let's first apply P = (2, 3, 4, 1) 

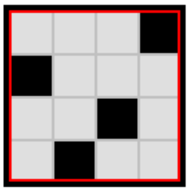
1 -> 2
2 -> 3
3 -> 4
4 -> 1

Then apply Q = (4, 3, 1, 2) 

1 -> 2 -> 3
2 -> 3 -> 1
3 -> 4 -> 2

4 -> 1 -> 4

We thus obtain the permutation

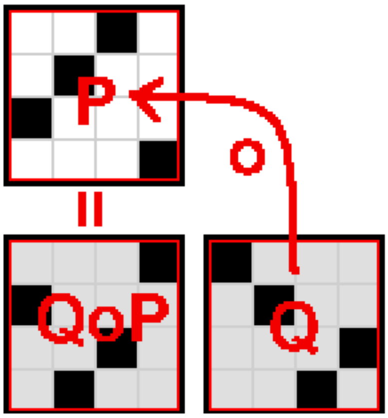


Q o P = (3, 1, 2, 4)

(1, 2, 3, 4)
P = (2, 3, 4, 1) Then apply Q = (4, 3, 1, 2)
Q o P = (3, 1, 2, 4)

Q o P = (1, 3, 4, 2)

Let's do the calculation graphically with the drawdown formula.



With the fabric we therefore have a permutation calculation tool, in graphical form.

How many different permutations of 4 elements are there ?

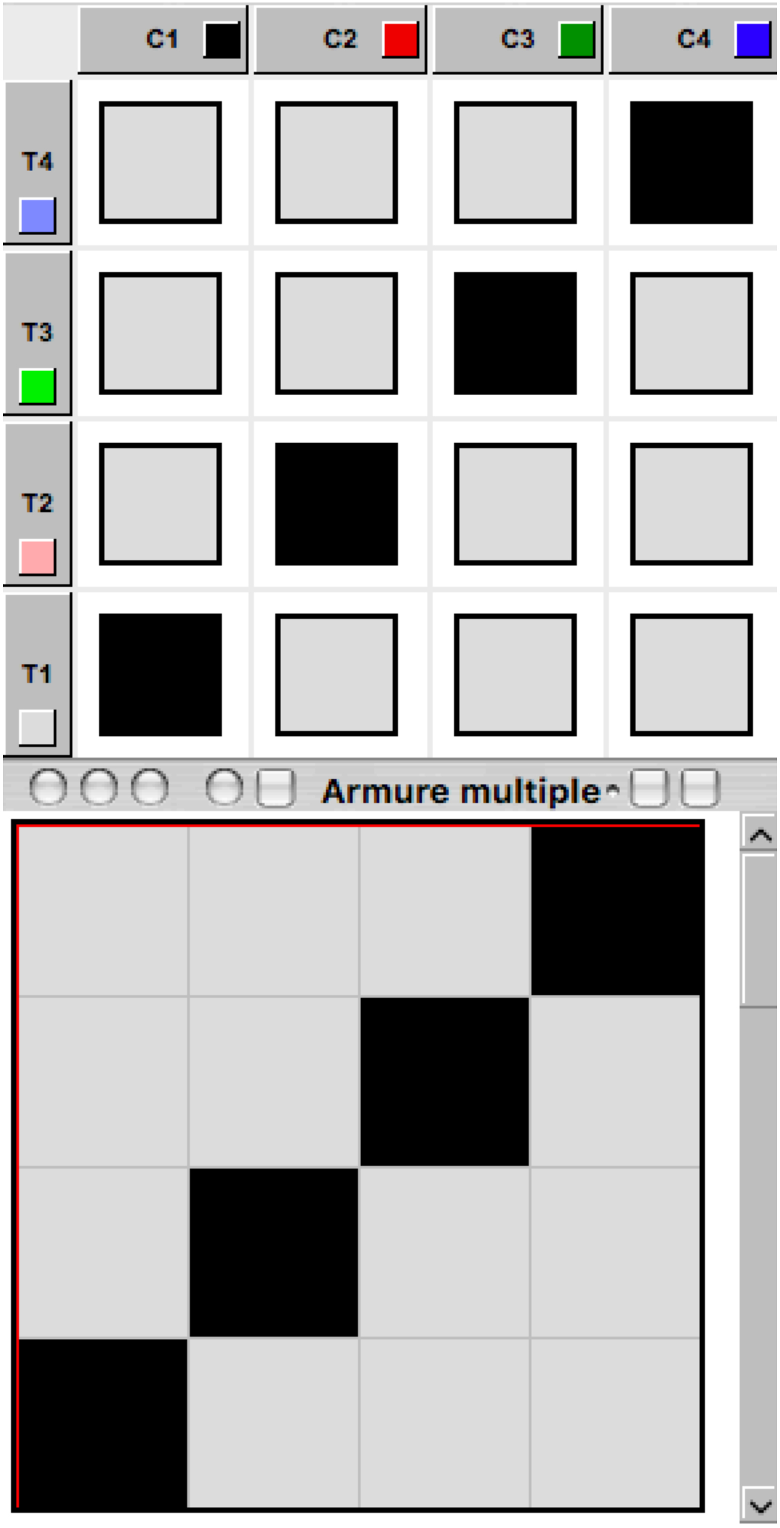
For the first shaft there are 4 possible choices.
For the second there are 3 possible choices.
For the third there are 2 possible choices.
For the fourth there is only one choice.

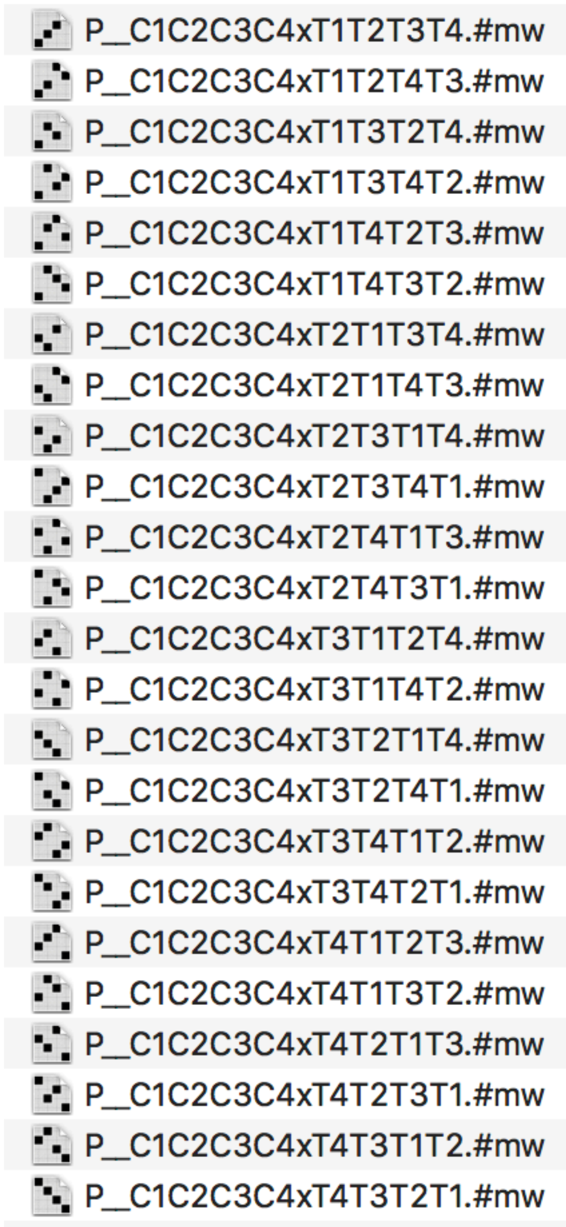
In the end the number of different permutations is:
4 x 3 x 2 x 1 = 24

What is note factorial 4 in mathematics : 4!
There are 4! = 24

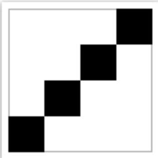
To generate them automatically we will use multiple weave structure.

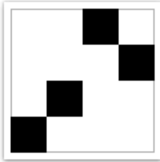
Just build twill weave 1/3 on 4 warps and 4 wefts.





Select the 4 wefts and choose the "Create structures by permutations" item from the "Multiple structure" menu and the 24 permutations will be saved to a folder in the form of multiple structures.


P_1234.tif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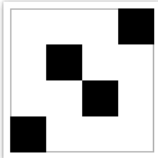
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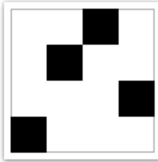
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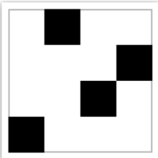
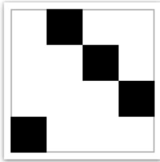

P_1423.tif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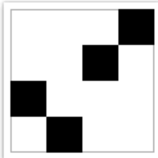
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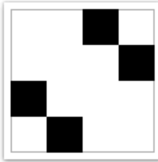
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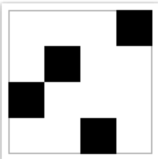
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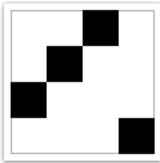
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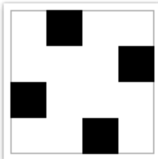
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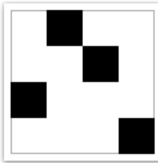
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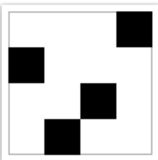
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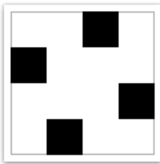
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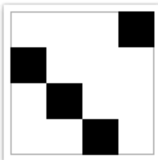
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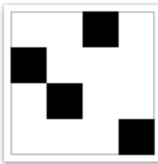
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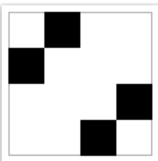
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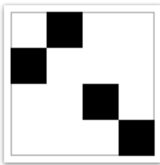
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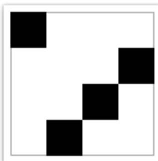
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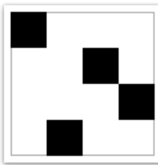
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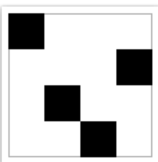
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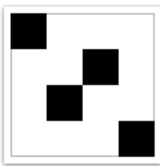
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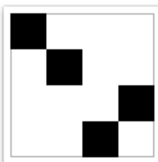
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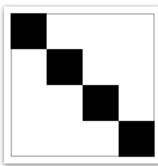
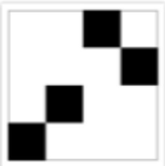

P_4321.tif

To find our bearings, we will classify and name them.

The identity i
Do not change the order of shafts


P_1234.tif

The 6 transpositions: the permutations that exchange two shafts.
The other shafts remain in their place (2 points on the 1st diagonal).
All permutations can be expressed as a product of transpositions. This means that we can do a mix of shafts by doing a series of shafts exchanges two by two.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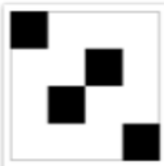
P_1324.tif


P_1432.tif

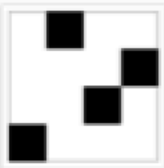

P_2134.tif


P_3214.tif


P_4231.tif

The 8 permutations that leave invariant a single shaft (only 1 point on the 1st diagonal).


P_1342.tif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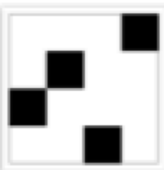
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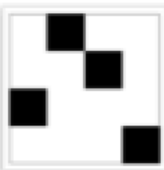
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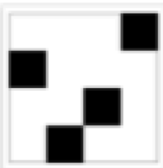
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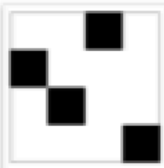
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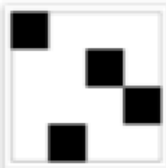
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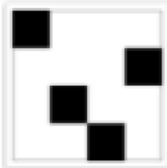
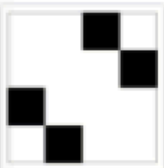

P_4213.tif

The 9 permutations that change all the shafts (which leave no invariant shaft).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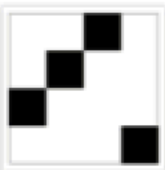
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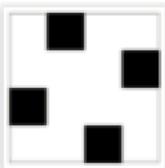
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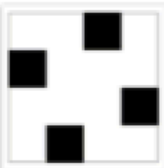
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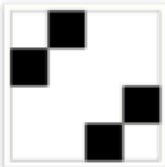
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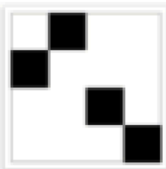
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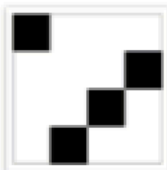
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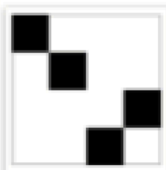
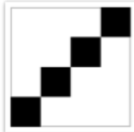
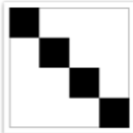

P_4312.tif


P_4321.tif

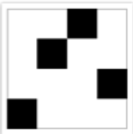
the last of which, -I, reverses the order of the shafts.

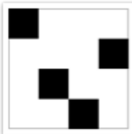
We will use the notations introduced  p 88

1) We note i the identical permutation that does not change the order of the shafts : i = 
P_1234.tif

2) We note -i the permutation that reverses the order of shafts : -i = 
P_4321.tif

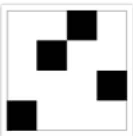
3) If p is a permutation, we write -p the permutation -i o p : -i o p = -p

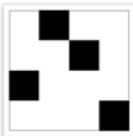
p = 
P_1342.tif

and -p = 
P_4213.tif

-p is the symmetric of p with respect to the horizontal.

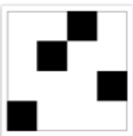
4) If p is a permutation, we denote p- the permutation p o -i : p o -i = p-
Note that in general -p ≠ p-

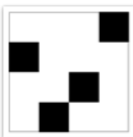
p = 
P_1342.tif

and p- = 
P_2431.tif

-p is the symmetric of p with respect to the vertical.

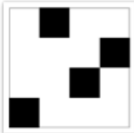
5) -(p-) = (-p)- = -p-

p = 
P_1342.tif

and -p- = 
P_3124.tif

-p- is the 180 ° rotation of p

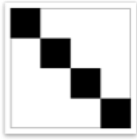
6) p^{-1} is the inverse of p : $p^{-1} o\ p = p^{-1}\ o\ p = i$

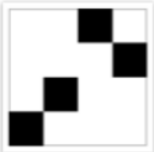
p =  P_1342.tif and p^{-1} =  P_1423.tif

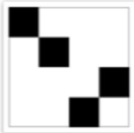
p^{-1} is the symmetrical of p with respect to the first diagonal.

With these notations we will be able to consider the 24 permutations as composed of 7 permutations: i, a, b, c, d, e, f

$i = i^{-1} =$ 
P_1234.tif

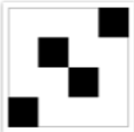
$-i = i- = (-i)^{-1} = (i-)^{-1} =$ 
P_4321.tif

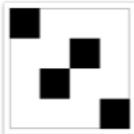
$a = a^{-1} =$ 
P_1243.tif

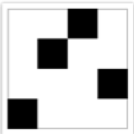
$-a =$ 
P_4312.tif

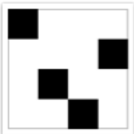
$a- =$ 
P_3421.tif

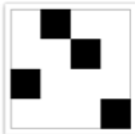
$-a- =$ 
P_2134.tif

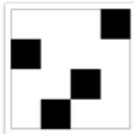
$b = b^{-1} = -b- =$ 
P_1324.tif

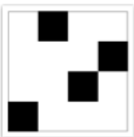
$-b = b- = (-b)^{-1} = (b-)^{-1} =$ 
P_4231.tif

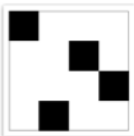
$c =$ 
P_1342.tif

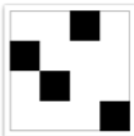
$-c =$ 
P_4213.tif

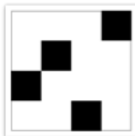
$c- =$ 
P_2431.tif

$-c- =$ 
P_3124.tif

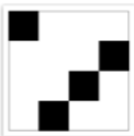
$c^{-1} =$ 
P_1423.tif

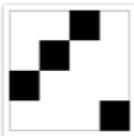
$-(c^{-1}) =$ 
P_4132.tif

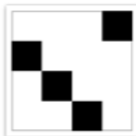
$(c^{-1})- =$ 
P_3241.tif

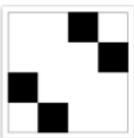
$-(c^{-1})- =$ 
P_2314.tif

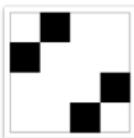
$d = d^{-1} =$ 
P_1432.tif

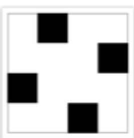
$-d =$ 
P_4123.tif

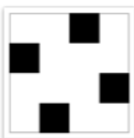
$d- =$ 
P_2341.tif

$-d- =$ 
P_3214.tif

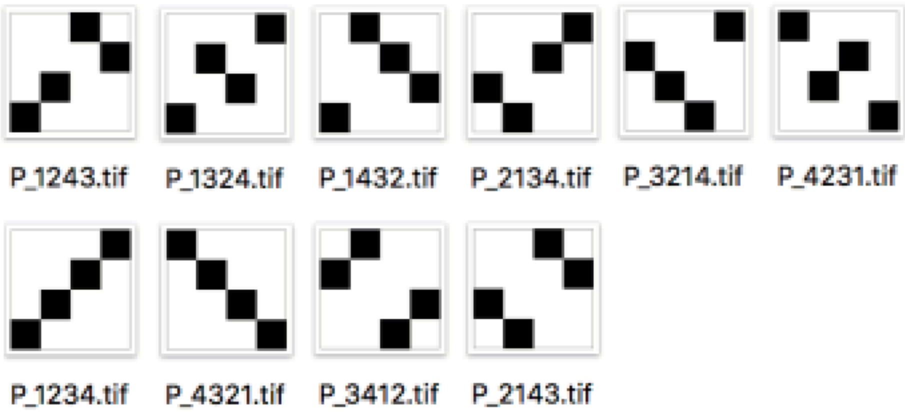
$e = e^{-1} = -e- =$ 
P_2143.tif

$-e = e- = (-e)^{-1} = (e-)^{-1} =$ 
P_3412.tif

$f = -f- =$ 
P_2413.tif

$-f = f- =$ 
P_3142.tif

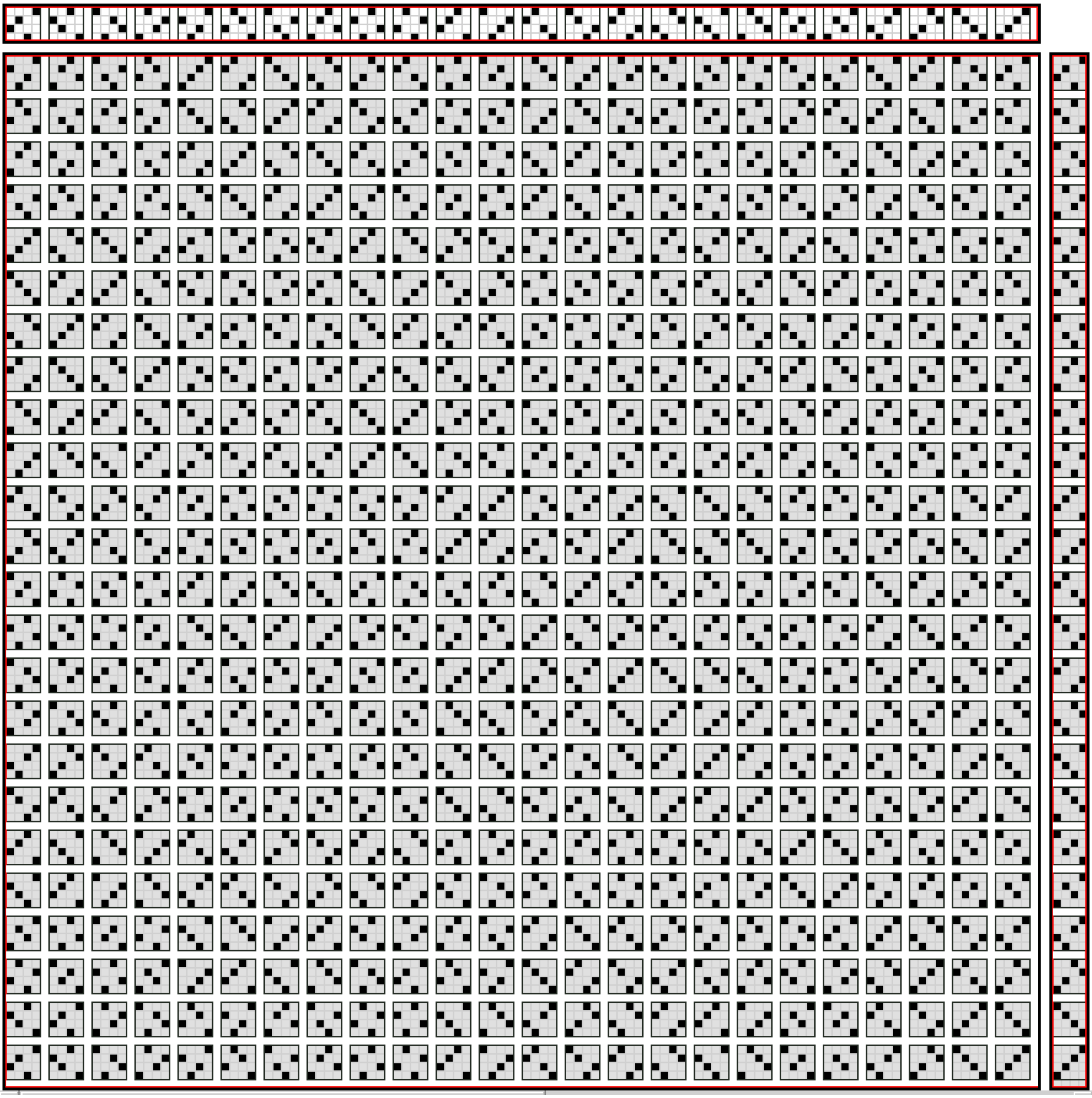
There are 10 involutional permutations (just symmetric because permutations are bijections).
The 6 transpositions plus i, -i and 2 other permutations
These permutations are symmetrical with respect to the 1st diagonal.
They are such that $p^2 = i$ and $p^{-1} = p$
If we apply twice the mix of shafts we find the initial order.



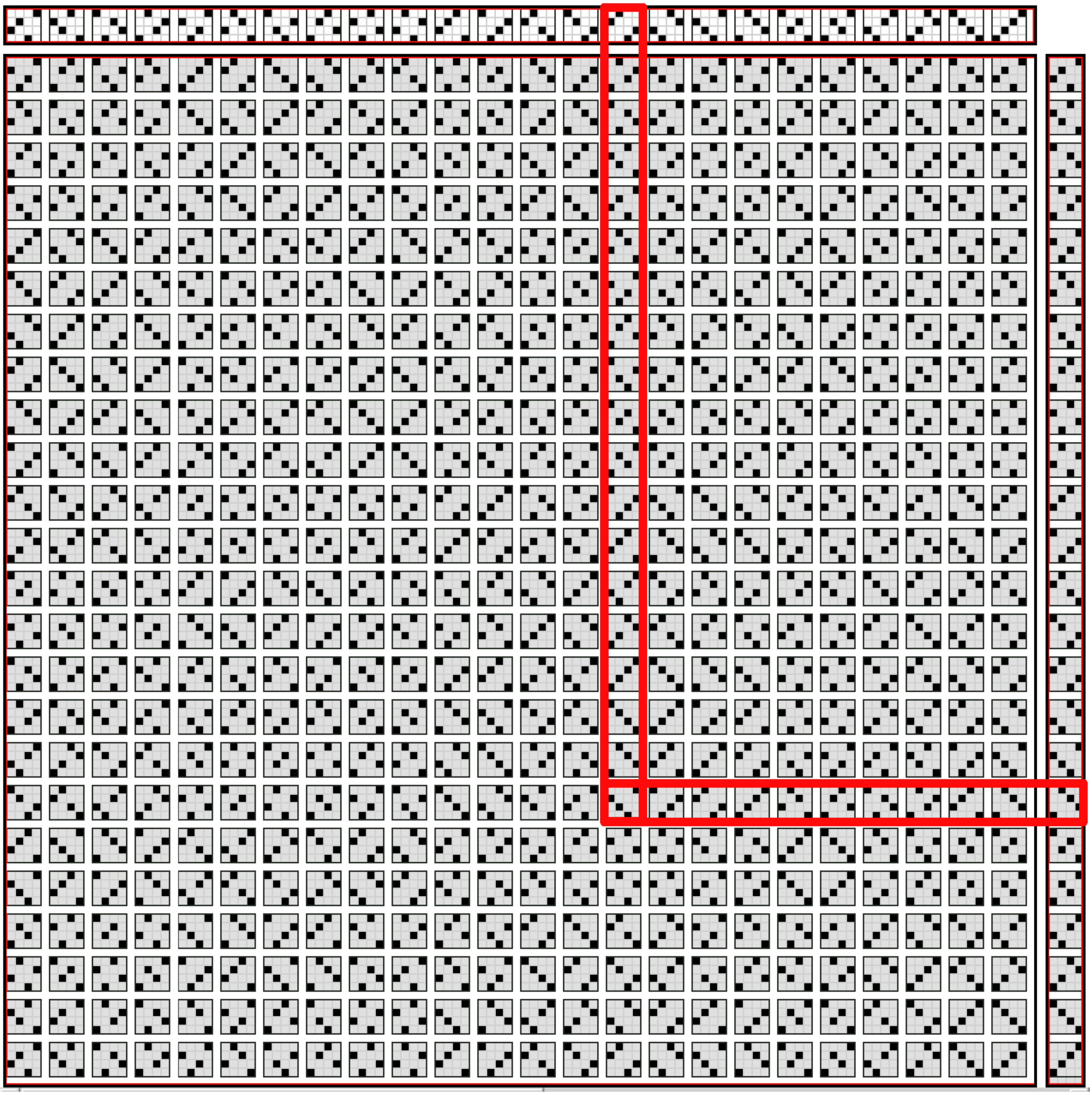
These have a particular interest for weaving. They make it possible to obtain all the symmetrical clothes with respect to the 1st diagonal of a threading with 4 blocks, only one block being lifted with each pick : see  p 83

To easily calculate permutations, we will use the drawdown formula.

I built a threading and a peg-plan with the 24 permutations together, so all possible calculations are done at once.



The multiplication table of permutations



To make the calculation, choose the permutation p in the peg-plan, choose the permutation q in the threading, at the intersection of the cardboard line and the tuck column is the result p o q. Attention to the order, the multiplication of the permutations is not commutative.

We see on the second diagonal of the drawdown at the bottom right the 10 involutive permutations whose square is equal to i.

Now we are going to look at the powers of the permutations pn

Mathematics brings us a very important result for weaving.

For every permutation p there exists a non-zero integer such that $p^n = i$
The smallest of these numbers is called the order of p

If we repeat a permutation several times, after a while we fall back on the identical permutation i.
If we mix the shafts of a threading with a permutation p, then if we mix it a second time with p, that is to say if we mix it with p^2, then with p^3, etc. we always ended up falling back on the initial threading. By mixing always with the same mix we end up not mixing anything !

The order of p will occur particularly in the threading constructions by amalgamation or telescoping.

Let's classify the 24 permutations 4 according to their order.

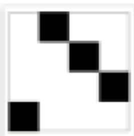
1) Ordre 1

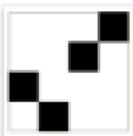
$i^1 = i$ P_1234.tif

1) Ordre 2 $p^2 = i$
9 permutations n = 2 : -i, a, -a-, b, -b, d, -d-, e, -e
Les 6 transpositions qui échange deux cadres, deux permutations qui échangent les cadres 2 à 2 et -i qui inverse l'ordre des cadres (qui échange aussi les cadres 2 à 2).

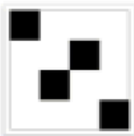

P_1243.tif


P_1324.tif

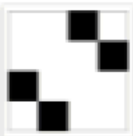

P_1432.tif


P_2134.tif


P_3214.tif

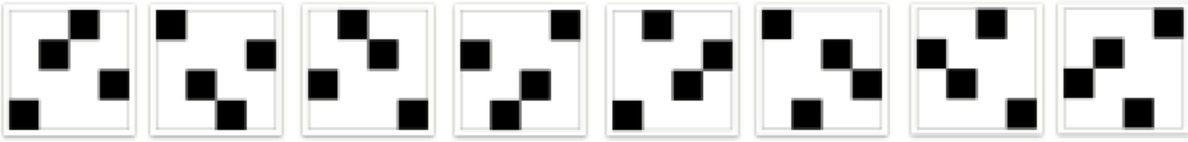

P_4231.tif


P_3412.tif


P_2143.tif


P_4321.tif

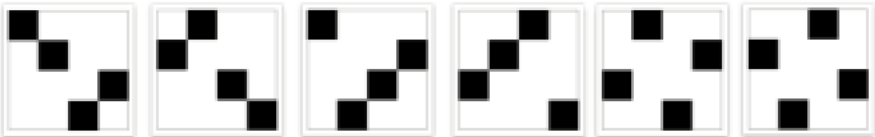
1) Ordre 3 $p^3 = i$
8 permutations avec n = 3 : c, -c, c-, -c-, c^{-1}, $-(c^{-1})$, $(c^{-1})-$, $-(c^{-1})-$
On retrouve les 8 permutations qui laissent invariant un seul cadre (1 seul point sur la 1ère diagonale).



P_1342.tif P_4213.tif P_2431.tif P_3124.tif P_1423.tif P_4132.tif P_3241.tif P_2314.tif

1) Ordre 4 $p^4 = i$

6 permutations avec n = 4 -a, a-, -d, d-, f et -f



P_4312.tif P_3421.tif P_4123.tif P_2341.tif P_2413.tif P_3142.tif

These results will be useful in the article [Telescoping and permutations]

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